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On the Fractal Dimension and Correlations in Percolation Theory

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We discuss the fractal dimension of the infinite cluster at the percolation threshold. Using sealing theory and renormalization group we present an explicit expression for the two-point correlation function within percolation clusters. The fractal dimension is given by direct integration of this function.

KEY WORDS: Fractals; percolation.

1. INTRODUCTION

One of the most intensively studied random fractals is the *percolating infinite cluster.*[1–5,4] Its popularity came from the fact that indeed percolation was shown to be a model which well describes inhomogeneous physical systems such as metal-insulator thin films,[6] gels,[7] or dilute magnetic systems.[8]

Much of the current interest in such systems concentrates on the influence of the geometrical structure on the physical properties in the vicinity of the percolation threshold, p_c.[6–10] As the concentration p approaches p_c, the pair connectedness length ξ diverges, $\xi \propto (p-p_c)^{-\nu}$. It is generally believed that on large length scales, $L \gg \xi$, the infinite cluster which appears for $p > p_c$ is homogeneous, with site (or bond) density $P_\infty \propto (p-p_c)^\beta \propto \xi^{-\beta/\nu}$. This homogeneity is believed to disappear for shorter length scales, $L < \xi$. For these scales, the infinite cluster is argued to be *self-similar*, with a typical *fractal dimensionality* D.[1–7,11,12] The value of D was discussed extensively in the literature.[1–7,11,12,13] To define D, consider a point on the infinite cluster, and count the number $M(L)$ of points on the

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same cluster within a volume L^d (of linear size L in d dimensions) centered at that point. The last condition is essential if we want to fulfill the Hausdorf–Besikovitz[3] definition of D. Self-similrity implies that[3,11,12]

$$M(L) \propto L^D, \qquad a \ll L \ll \xi \tag{1}$$

where a is a typical microscopic length.

For $L \gg \xi$, homogeneity implies that $M(L) \propto P_\infty L^d \propto \xi^{-\beta/\nu} \cdot L^d$. Assuming that ξ is the only relevant length in the problem, we may write $M(L, \xi)$ in the scaling form[11]

$$M(L, \ \xi) = \xi^{-\beta/\nu} \cdot L^d \cdot m\left(\frac{L}{\xi}\right) \tag{2}$$

For $L \ll \xi$, M should become independent of ξ. Thus $m(x) \propto x^{-\beta/\nu}$ and $M(L) \propto L^{d-\beta/\nu}$, i.e.,

$$D = d - \beta/\nu \tag{3}$$

This result also follows from finite size scaling at p_c,[5] and has been confirmed by independent measurements of D, β, and ν for two-dimensional percolation systems.[11]

It is the aim of this paper to discuss these relations. In particular, Section 2 exhibits a general self-consistent calculation for $M(L)$, in the self-similar regime. This calculation yields the result

$$D = (\beta + \gamma)/\nu \tag{4}$$

where γ describes the divergence of the mean cluster size. For $d < 6$, the hyperscaling relation $d\nu = 2\beta + \gamma$ yields the equivalence of Eqs. (3) and (4). As we show in Section 3, this is no longer the case for $d > 6$, when only (4) is correct, yielding $D \equiv 4$, nor at $d = 6$, when logarithmic corrections are found.

The breakdown of hyperscaling results from the existence of a "dangerous irrelevant variable," and leads to a generalized scaling form replacing Eq. (2). These discussed in Section 4.

2. SELF-CONSISTENT DERIVATION OF *D*

Consider the conditional probability $\rho_s(r)$ that a site at a distance r from the origin belongs to a cluster of s sites, given that the origin belongs to it.[14] We can express the percolation connectedness correlation function, $G(r)$, as an average over $\rho_s(r)$,

$$G(r) = \sum_{s=1}^{\infty} sn_s \rho_s(r) + P_\infty \rho_\infty(r) - P_\infty^2 \tag{5}$$

where sn_s is the probability that a site belongs to a finite cluster of s sites.

Let r_s be the typical linear size of a cluster of s sites. We expect $\rho_s(r)$ to decay exponentially for $r > r_s$. We shall thus use the approximate value $\rho_s(r) \simeq 0$ for $r > r_s$, and the sum in Eq. (5) will contain only sizes $s < s_r$, where s_r is the inverse function of r_s.

The function sn_s is known[2] to decay exponentially for $s > s_\xi$. In the same spirit as above, we approximate sn_s by zero for $s > s_\xi$. The sum in Eq. (5) thus contains only terms with $r < r_s < \xi$. For such length scales we expect all the clusters to have the same self-similar structure. Therefore, we write $\rho_s(r) = \rho_\infty(r)$.

Combining all these simplifying assumptions, Eq. (5) now becomes[14]

$$\rho_\infty(r) = [G(r) + P_\infty^2] \Big/ \left[\sum_{s_r}^{s_\xi} sn_s + P_\infty \right] \tag{6}$$

For $r \gg \xi$ one expects $G(r)$ to decay exponentially. The sum in the denominator of (6) is also vanishing, and we end up with $\rho_\infty(r) \simeq P_\infty$. This is the *homogeneous* regime.

For $r \lesssim \xi$, "strong" self-similarity[2] implies that $s_r \propto r^D$. Using also $sn_s \propto s^{1-\tau}$ $(s \lesssim s_\xi)$,[2] the sum in the denominator becomes of order $s_r^{2-\tau} \propto r^{-D(\tau-2)}$, which is expected to be large compared to P_∞. In the same range, we expect that $G(r) \propto r^{-(d-2+\eta)} \gg P_\infty^2$. Thus,

$$\rho_\infty(r) \propto r^{2-d-\eta+D(\tau-2)}, \qquad r \ll \xi \tag{7}$$

The "mass" on the infinite cluster within a volume L^d around the (occupied) origin is thus $(L < \xi)$

$$M(L) = \int^L d^d r \, \rho_\infty(r) \propto L^{2-\eta+D(\tau-2)} \tag{8}$$

Comparison with Eq. (1) now yields

$$D = \frac{2-\eta}{3-\tau} = \frac{\gamma+\beta}{\nu} \tag{9}$$

where on the right-hand side we used[2] $\gamma = (3-\tau)/\sigma$, $\sigma = 1/(\gamma+\beta)$ and $\gamma = (2-\eta)\nu$. This is our Eq. (4), derived without any hyperscaling relations.

In the following sections we summarize existing and new expressions for sn_s and for $G(r)$, and use them to derive $p_\infty(r)$ and $M(L)$ explicitly.

3. EXPLICIT RESULTS

The explicit calculations of sn_s and of $G(r)$ are based on the mapping of the percolation problem on the limit $q \to 1$ of the q-state Potts model. The Hamiltonian of this model is written[15]

$$\mathscr{H} = -\frac{1}{4}\int (r_0 + k^2) \sum_{i=1}^{q} Q_{ii}(\mathbf{k})\, Q_{ii}(-\mathbf{k}) + w \iint \sum_i Q_{ii}(\mathbf{k})\, Q_{ii}(\mathbf{k}'')\, Q_{ii}(-\mathbf{k}-\mathbf{k}''), \tag{10}$$

with r_0 linear in $(p_c - p)$. The upper critical dimension of the model is $d_u = 6$.[16] The renormalization group (RG) recursion relations are[17]

$$\frac{dr}{dl} = (2+\eta)r + O(w) \tag{11}$$

$$\frac{dw}{dl} = \left(\frac{\varepsilon}{2} - \frac{3}{2}\right) w + O(w^3) \tag{12}$$

where $\varepsilon = 6 - d$, $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$ and

$$\eta = -48 K_d w^2 \tag{13}$$

For $d < 6$, $w(l)$ flows to a fixed point, with $(w^*)^2 = O(\varepsilon)$. One can then add an ordering "ghost" field h, derive an equation of state $Q(h)$,[17] and Laplace-transform it to obtain sn_s. Following Stephen,[18] this yields

$$sn_s = \frac{1}{(48\pi wc)^{1/2}} s^{-(3/2-\varepsilon/14)} \exp\left(-\frac{|t|^2 s}{48wc}\right) \times \left[1 \pm \frac{1}{7}\varepsilon\left(\frac{\pi |t|^2 s}{48wc}\right)^{1/2}\right] + O(\varepsilon^2), \qquad d < 6 \tag{14}$$

where $t = (p_c - p)/p_c$ and c is a constant.

For $d > 6$ the behavior is characterized by the Gaussian fixed point, $r^* = w^* = 0$, in the vicinity of which one has

$$r(l) = r(0)e^{2l}, \qquad w(l) = w(0)e^{(3-d/2)l} \tag{15}$$

Repeating the same calculation we rederive the mean field result[18]

$$sn_s = \frac{1}{(48\pi wc)^{1/2}} s^{-3/2} \exp\left(-\frac{|t|^2 s}{48wc}\right), \qquad d > 6 \tag{16}$$

At $d = 6$ the flow to the Gaussian fixed point is slower, $w(l) \propto w(0)/\sqrt{l}$. This implies that $t(l) = t(0)e^{2l}/l^{5/21}$, and introduces additional powers of l into various expressions.[17] When $t(l) = O(1)$ these l's are replaced by logarithmic factors, e.g., $\ln|t/t_0|$. Finally, the same calculation yields[19]

$$sn_s \propto w^{4/7} \left[\ln \frac{s|t_0|^2}{48wc}\right]^{2/7} s^{-3/2} \exp\left(-\frac{|t|^2 s}{48wc}\right), \qquad d = 6 \tag{17}$$

where t_0 is a constant. This result for sn_s is reported here for the first time.

We now turn to the calculation of $G(r)$. The Fourier transform of $G(r)$ has the scaling form[20]

$$\hat{G}(\mathbf{k}, r, w) = \exp\left[2l - \int_0^l \eta(l')\, dl'\right] G(e^l\mathbf{k}, r(l), w(l)) \tag{18}$$

One may obtain $\hat{G}$ by iterating the RG recursion relations until $t(l) + e^{2l}k^2 = 1$, and then using perturbation theory.[21]

At $p = p_c$, i.e., $t = 0$, we indeed confirm that

$$\hat{G}(\mathbf{k}, 0, w)^{-1} \propto k^{2-\eta}, \qquad d < 6 \tag{19}$$

with $\eta = -\varepsilon/21$.

For $d > 6$ one obtains the Gaussian result,

$$\hat{G}(\mathbf{k}, 0, w)^{-1} = k^2, \qquad d > 6 \tag{20}$$

and at $d = 6$ one has the new result

$$\hat{G}(\mathbf{k}, 0, w)^{-1} \propto k^2[\ln(k/k_0)]^{-1/21} \tag{21}$$

Note that such logarithmic factors in $\hat{G}$ are expected whenever η is of order ε!

We are now ready to combine sn_s and $G(r)$ to derive $\rho_\infty(r)$. For $d < 6$, at $t = 0$, Eq. (14) is clearly of the form $sn_s \propto s^{1-\tau}$, with $\tau = 5/2 - \varepsilon/14$. Similarly, $G(r)$ is the Fourier transform of Eq. (19), $G(r) \propto r^{-(d-2+\eta)}$. Substitution into Eq. (8) indeed confirms Eq. (1), with

$$D = 4 - \frac{10}{21}\varepsilon + O(\varepsilon^2) \tag{22}$$

This also agrees with the hyperscaling result (3).

Table I. Results for $\rho_\infty(r)$ and $M(L)$

	Self-similar regime			Homogeneous regime		
	$d=6-\varepsilon$	$d=6$	$d>6$	$d=6-\varepsilon$	$d=6$	$d>6$
$\rho_\infty(r)$	$r^{-[2-(11/21)\varepsilon]}$	$w(\ln r)^{-10/21}r^{-2}$	wr^{4-d}	$\xi^{-[2-(11/21)\varepsilon]}$	$w^{-1}\xi^{-2}(\ln\xi)^{11/21}$	$w^{-1}\xi^{-2}$
$M(L)$	$L^{4-(10/21)\varepsilon}$	$w(\ln L)^{-10/21}L^4$	wL^4		$L^d\rho_\infty$	

For $d>6$, Eq. (15) shows that $w(l)$ decays to zero as $l\to\infty$. However, w appears in denominators of various expressions, e.g., Eq. (16). One can therefore not set $w=w^*=0$. Such variables are called "dangerously irrelevant."[22] The calculation of Section 2 can still be repeated, if one substitutes $s_r \propto w^x r^D$, $M(L)\propto w^x L^D$, $sn_s \propto w^{-1/2}s^{-3/2}$, $G(r)\propto r^{-(d-2)}$. One then finds[14] $x=1$ and $D=4$ for all $d>6$. Clearly, this agrees with Eq. (4) (with $\beta=\gamma=2\nu=1$), but not with the hyperscaling result (3).

At $d=6$ we substitute $s_r\propto w^x(\ln r)^y r^D$, and identify $x=1$, $y=-10/21$, $d=4$.

In the homogeneous regime, $r\gg\xi$, we confirm explicitly that $\rho_\infty=P_\infty$. Our results are summarized in Table I.

4. MODIFIED SCALING FOR $d>6$

For $d>6$, we concluded that one should keep track of explicit dependences on w. Thus, Eq. (2) must now be replaced by

$$M(L,\xi,w)=P_\infty L^d\tilde{m}\left(\frac{L}{\xi},\xi^{3-d/2}w\right) \tag{23}$$

where the form $\xi^{3-d/2}w$ results from Eq. (15) (used until $e^l=\xi$). Substituting $P_\infty\propto 1/w\,\xi^2$, this becomes

$$M(L,\xi,w)=\frac{L^d}{w\xi^2}\tilde{m}\left(\frac{L}{\xi},\xi^{3-d/2}w\right) \tag{24}$$

The function $\tilde{m}$ depends singularly on its second variable: when $L\ll\xi$, $\tilde{m}(x,y)\propto x^{4-d}y^2$, yielding $M\propto wL^4$ as required.

One may interpret the additional variable in Eq. (24) as introducing a new length, $L_w=w^{2/(d-6)}$. This length may be associated with the size of "blobs" of bonds on the infinite cluster,[4] since w is associated with the probability of three-bond vertices.[14] The behavior of M now depends on both L/ξ and ξ/L_w.

For $d < 6$, the crossover from the homogeneous to the self-similar regime occurs at $L \sim \xi$. For $d > 6$, the appearance of L_w defines a series of crossover lengths,[14]

$$L_k = (L_w^{d-6} \xi^{2k})^{1/(d-6+2k)} \tag{25}$$

The two terms in the numerator of Eq. (6) become comparable at L_2, the two limiting behaviors of $M(L)$ become comparable at L_1 and those of $g(L)$ where g is the conductance at scale L[14,19] become comparable at L_3. There probably exists a range of length scales, below ξ, through which various physical quantities cross over from their self-similar to their homogeneous behavior. Clearly, all the physical properties scale according to our self-similar predictions (e.g., $M \propto wL^4$, $g \propto L^{-2}$) for $L < L_1 \propto \xi^{2/(d-4)}$, and according to the homogeneous ones for $L > \xi$. It is not yet clear to us whether the range $L_1 < L < \xi$ represents a third scaling regime, or whether there is a separate crossover for each property. One would also like to obtain a geometrical interpretation of the lengths L_k.

For $d = 6$, the two limiting expressions become comparable at

$$L_0 \simeq w\xi(\ln \xi)^{-1/2} < \xi \tag{26}$$

In this case, the second argument in Eq. (23) is replaced by $w/\ln \xi$ (or by $w/\ln L$), and the simple scaling form (2) is again violated.

NOTE ADDED IN PROOF

For $d < 8$, finite $(p - p_c) < 0$, and sufficiently large n one expects a crossover from Eq. (16) to the distribution function of lattice animals [A. B. Harris and T. C. Lubensky, *Phys. Rev. B* **24**:2656 (1981)]. This should not affect the scaling properties of averages of powers of n calculated with (16), nor our results at $p = p_c$. We are grateful to A. B. Harris for discussions of this point.

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